

# Quantization of Scalar Field Theory with Internal Symmetry

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## Abstract

We consider simple theoretical model of scalar fields in one spatial dimension with an internal symmetry. We use the Schrödinger picture to describe the quantum properties of localized solutions. Making use of collective coordinates method allows to develop a perturbation theory, which exactly describes symmetry properties of theory. As examples the  $U(1)$  and  $SU(2)$  symmetries are analyzed and dependence of the energy of quantum levels on quantum numbers caused by corresponding symmetries is obtained.

## 1 Introduction

Quantization of localized in space solutions has a long history. There are several approaches to this problem, mainly within the framework of functional integration[1]-[7]. The method of collective coordinates, which indeed are the parameters of a symmetry group, is one of them. We apply in present investigation the operator approach of this method for solving the Schrödinger equation. The advantage of this approach is the absence of an ambiguity related with the operator ordering. Besides a perturbation theory can be constructed that is manifestly invariant with respect to the group of the symmetry of a theory under consideration in every order of the perturbation series.

We consider below the simple model with internal symmetry, which is of interest even today, although many articles were devoted to this issue[2],[8]-[10]. The point is that it is possible in theories with internal symmetry to obtain stable field configurations in more than one space dimension. In papers[2, 8, 10] the charge symmetry has been studied and contribution of the charge depending terms into the energy of the system has been evaluated. We want to focus our attention on[8]. In this article the charge symmetry is investigated in detail. It has been shown, that there exist charged classical field configurations that are

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stable owing to the existence of a charge. The centrifugal term and its contribution to the energy have been calculated in the framework of the functional integration. The charge is assumed to take on integer values. The extension of the method to the case of  $SU(2)$  symmetry is discussed. The approach used in the present article starts from the quantum theory and suggests the construction of a perturbation theory around the classical neutral field configuration (around charged configurations is also possible). The charge (or the isotopic spin) is quantized such that the exact dependence of the energy on the charge (or the isotopic spin) is represented. Besides the operator ordering problem does not arise in this approach.

## 2 The model with the $U(1)$ symmetry

We consider the model of scalar triplets in one space dimension with a broken  $SU(2)$  symmetry. The Hamiltonian of the model is

$$H = \int dx \left\{ \frac{1}{2} \pi_\alpha(x) \pi_\alpha(x) + \frac{1}{2} \frac{\partial \varphi_\alpha(x)}{\partial x} \frac{\partial \varphi_\alpha(x)}{\partial x} + U(\varphi_i(x), \varphi_3(x), g) \right\} \quad (1)$$

with the following commutation relations between the field operators:

$$[\pi_\alpha(x), \pi_\beta(y)] = i \delta_{\alpha\beta} \delta(x - y). \quad (2)$$

the Greek indices  $\alpha, \beta, \gamma \dots$  take on values 1, 2, 3 and Latin indices - 1, 2. The fields  $\varphi_i(x)$  form the charged fields. The potential  $U(\varphi_i(x), \varphi_3(x), g)$  is suggested to be breaking the  $SU(2)$  symmetry and obeys the condition:

$$U(\varphi_i(x), \varphi_3(x), g) = g^2 U(g\varphi_i(x), g\varphi_3(x), 1). \quad (3)$$

Thus the model is  $U(1)$  invariant and as a result the charge of the system

$$Q = \int dx \{ \varphi_1(x) \pi_2(x) - \varphi_2(x) \pi_1(x) \} \quad (4)$$

is conserved. We apply the method of collective coordinates to the Schrödinger equation

$$H\Psi(\varphi_i(x), \varphi_3(x)) = E\Psi(\varphi_i(x), \varphi_3(x)). \quad (5)$$

mention that the operator  $\pi_\alpha(x)$  is considered as the functional derivative  $\delta/i\delta\varphi_\alpha(x)$ .

Let us introduce the transformation:

$$\varphi_i(x) = D_{ij}(\vartheta)(gu(x)\delta_{j1} + \Phi_j(x)), \quad \varphi_3(x) = g\sigma(x) + \Phi_3(x), \quad (6)$$

in which the parameter  $\vartheta$  together with  $\Phi_\alpha(x)$  compose the new set of operators and the matrix  $D(\vartheta)$  is the known two dimensional matrix of a rotation:

$$D = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \quad (7)$$

In order to retain the total number of independent variables a subsidiary condition has to be imposed, namely

$$\int dx N(x) \Phi_2(x) = 0. \quad (8)$$

The c-number  $N(x)$  is normalized as follows:

$$\int dx N(x) u(x) = 1. \quad (9)$$

We should now express the momenta  $\pi_\beta(y)$  in terms of the new variables  $\theta$ ,  $\Phi_\alpha(x)$ . This is done according to the ordinary rules of differentiation provided that the subsidiary condition (8) will be taken into account. In addition to (8) we need to introduce the following projection operator:  $A_{ij}(x, y) = \delta_{ij}(x - y) - \delta_{i2}\delta_{j2}N(x)u(x)$  with the properties:

$$\int dy A_{i2}(x, y) N(x) = \int dx u(x) A_{2j}(x, y) = 0. \quad (10)$$

The fields  $\Phi_i(x)$  can be expressed as

$$\Phi_k(y) = \int dz A_{kj}(y, z) \overline{D}(\vartheta)_{ji}(\vartheta) (\varphi_i(z) - \delta_{i1}u(z)), \quad (11)$$

in which  $\overline{D}(\vartheta)D = I$ . Taking the functional derivative of (8) and (11) with respect to  $\varphi_i(x)$ , one obtains:

$$\pi_i(x) = \overline{D}_{ji}(\vartheta) \{ \Pi_k(x) + \frac{N(x)\delta_{2k}}{(\quad)} g - F(p_\vartheta + \overline{Q}) \} \quad (12)$$

$$\pi_3(x) \equiv \Pi_3(x) = \frac{\delta}{i\delta\Phi_3(x)} \quad (13)$$

The  $U(1)$  symmetry is the simplest internal symmetry. In this expression the following notations are used:

$$\overline{Q} = \overline{J}_{ik} \int dx \Phi_k(x) \Pi_i(x),$$

and is the charge of mesons (described by fields  $\Phi_k(x)$ ), the antisymmetric matrix

$$\overline{J}_{ik} = \frac{d\overline{D}_{ik}}{d\vartheta} |_{\theta=0}$$

and is the element of the algebra of the group  $U(1)$ ,  $F = \int dx N(x) \Phi_1(x)$ . For the new set of the variables of the system the following commutation relations hold:

$$[\vartheta, p_\vartheta] = i, [\Phi_i(x), \Pi_k(y)] = iA_{ik}(x, y), [\Phi_3(x), \Pi_3(y)] = i\delta(x - y). \quad (14)$$

We now can write in the new representation the kinetic energy:

$$\begin{aligned}
K &= \frac{1}{2} \int dx \pi_a(x) \pi_a(x) = \\
&\frac{1}{2} \int dx \{ \Pi_3^2(x) + [\Pi_k(x) + \frac{N(x)\delta_{k2}}{g-F}(p_\vartheta + \overline{Q})][\Pi_k(x) + \\
&\frac{N(x)\delta_{k2}}{g-F}(p_\vartheta + \overline{Q})] - i \frac{N(x)\Pi_1(x)}{g-F} \}
\end{aligned} \tag{15}$$

As we can see the kinetic energy does not contain the variable  $\vartheta$ . It is easy to verify that in the new representation the charge operator  $Q$  is reduced to  $p_\vartheta$  :  $Q = p_\vartheta$ . The equality (3) and the transformation (6) make it possible to expand the potential  $U$  as series in inverse powers of  $g$ , namely

$$\begin{aligned}
U &= g^2 U(u(x), \sigma(x)) + g \frac{\partial U(u(x), \sigma(x))}{\partial u(x)} + \frac{\partial U(u(x), \sigma(x))}{\partial \sigma(x)} + \\
&\frac{1}{2} \frac{\partial^2 U(u(x), \sigma(x))}{\partial u(x)^2} + \frac{1}{2} \frac{\partial^2 U(u(x), \sigma(x))}{\partial \sigma(x)^2} + \frac{\partial^2 U(u(x), \sigma(x))}{\partial \sigma(x) \partial u(x)} + \dots
\end{aligned} \tag{16}$$

Thus the variable  $\vartheta$  is cyclic and the operator  $p_\vartheta$  can be replaced with the c-number. Before doing this, we introduce the transformations, which eliminate the linear and cross terms in the operators  $\Pi_i$  from the kinetic energy. Firstly we assume  $N(x) = \lambda u(x)$  such that owing to (9)

$$\lambda \int dx u^2(x) = 1.$$

The quantity  $F$  can now be rewritten as  $\lambda h$  with  $h = \int dx u(x) \Phi_1(x)$ . Secondly we change the wave functional  $\Psi(\vartheta, \Phi_\alpha(x))$  as follows:

$$\Psi(\vartheta, \Phi_\alpha(x)) = \frac{1}{\sqrt{g - \lambda h}} \tilde{\Psi}(\vartheta, \Phi_\alpha(x)).$$

Taking functional derivative with respect to  $\Pi_1(x)$  one obtains for the Hamiltonian of the system

$$\begin{aligned}
H &= \int dx \{ + \frac{1}{2} \frac{\partial}{\partial x} (gu(x)\delta_{i1} + \Phi_i(x)) \frac{\partial}{\partial x} (gu(x)\delta_{i1} + \Phi_i(x)) + \\
&\frac{\partial}{\partial x} (g\sigma(x) + \Phi_3(x)) \frac{\partial}{\partial x} (g\sigma(x) + \Phi_3(x)) + U + \frac{1}{2} \Pi_\alpha(x) \Pi_\alpha(x) \} + \\
&\frac{\lambda}{2} [ \frac{1}{g - \lambda h} (p_\vartheta + \overline{Q}) \frac{1}{g - \lambda h} (p_\vartheta + \overline{Q}) - \frac{1}{4} ]
\end{aligned} \tag{17}$$

which acts now on  $\tilde{\Psi}(\vartheta, \Phi_\alpha(x))$ . The operator  $U$  is assumed to be expanded as in (16). We can now factor out the  $\vartheta$ -dependence in the wave functional since  $\vartheta$  is cyclic:

$$\tilde{\Psi}(\vartheta, \Phi_\alpha(x)) = \exp(im\vartheta) \Psi'(\Phi_\alpha(x))$$

with  $m = 0, \pm 1, \pm 2, \pm 3 \dots$  replacing the operator  $p_\vartheta$  with  $m$ . All the calculations are so far accurate and we did not make any approximation. We have eliminated the  $\vartheta$ -dependence from the Hamiltonian and it is now the function of the quantum number  $m$ , thereby the charge has been quantized. One can now construct a perturbation theory, which would be manifestly  $U(1)$ -invariant in every order, by means of expanding the Hamiltonian, the energy and the  $\Psi'(\Phi_a(x))$  in terms of inverse powers of  $g$  as follows:

$$\begin{aligned} H &= g^2 H_0 + g H_1 + H_2 + g^{-1} H_3 + g^{-2} H_4 + \dots \\ E &= g^2 E_0 + g E_1 + E_2 + g^{-1} E_3 + g^{-2} E_4 + \dots \\ \Psi'(\Phi_\alpha(x)) &= \Psi_0 + g^{-1} \Psi_1 + g^{-2} \Psi_2 + \dots \end{aligned} \quad (18)$$

We next solve the system of equations:

$$\begin{aligned} (H_0 - E_0)\Psi_0 &= 0, \\ (H_0 - E_0)\Psi_1 + (H_1 - E_1)\Psi_0 &= 0, \\ (H_0 - E_0)\Psi_2 + (H_1 - E_1)\Psi_1 + (H_2 - E_2)\Psi_0 &= 0, \\ &\dots \end{aligned} \quad (19)$$

We do not specify the potential  $U$  since we aimed to quantize the charge and to obtain the exact dependence of the Hamiltonian on the charge. We only make some remarks regarding the equations (19). It is evident, that the leading equations of order  $g^2$  and  $g$  reproduce the classical equations of motion giving zero charge solutions for  $u(x)$  and  $\sigma(x)$  and classical energy of neutral field configurations. The equation of order  $g^{-2}$  is bilinear in field operators  $\Phi_\alpha(x)$  and  $\Pi_\alpha(x)$  that formally can be diagonalized. The result is infinite sum of oscillators, which must be regularized. The charge dependence of the energy arises in the approximation of the order  $g^{-2}$  and is of the form:

$$E_4 = \frac{\lambda}{2}(m^2 + 2m \langle \overline{Q} \rangle_0 + \langle \overline{Q}^2 \rangle - \frac{1}{4}).$$

The symbol  $\langle \rangle$  denotes the average over the ground states of oscillators.

### 3 $SU(2)$ symmetry

In this section we apply the method to the same model, but with  $SU(2)$  symmetry. The fields  $\varphi_\alpha(x)$  are now assumed to belong to the adjoint representation of  $SU(2)$ . The corresponding conserved quantity is an isotopic spin, the third component of which defines the charge of a system:

$$t_\alpha = \varepsilon_{\alpha\beta\gamma} \int dx \varphi_\beta(x) \pi_\gamma(x). \quad (20)$$

The transformation which introduces collective coordinates is similar to (6) but with  $D(\theta, \phi)$  depending on two parameters:

$$\varphi_\alpha(x) = D_{\alpha\beta}(\theta, \phi) \{u(x) \delta_{\beta 3} + \Phi_\beta(x)\}. \quad (21)$$

So the phase space is now extended by two additional variables and together with the fields  $\Phi_\alpha(x)$  they form the new set of variables. The subsidiary conditions and the projection operator needed for evaluating the momentums in the new representation are:

$$\int dx N_{ik} \Phi_k(x) = 0, \quad (22)$$

$$A_{ij} = \delta_{ij} \delta(x-y) - N_{ik}(x) M_{kj}(y). \quad (23)$$

The quantity  $N_{ik}$  can be chosen to satisfy the condition

$$\int dx N_{ik}(x) M_{kj}(x) = 1. \quad (24)$$

Omitting the details of calculation, we give the final expression for momentums  $\pi_\alpha(x)$ :

$$\pi_\alpha(x) = \overline{D}_{\beta\alpha}(\theta, \phi) \{ \Pi_\beta(x) - (g + F)_{j\sigma}^{-1} N(x)_{ji} \delta_{\beta i} [\overline{t}_\sigma + T_\sigma] \} \quad (25)$$

in which the following notations are used<sup>2</sup> :

$$F_{\sigma j} = \int dx (\overline{J}_\sigma)_{i\nu} N_{ji}(x) \Phi_\nu(x), \quad (26)$$

$$\delta_{\sigma m} M_{im}(x) = \delta_{\sigma m} (\overline{J}_m)_{i3} u(x), \quad (27)$$

$$\overline{T}_\sigma = (\overline{J}_\sigma)_{i\nu} \int dx \Phi_\nu(x) \Pi_i(x), \quad (28)$$

The matrices  $(\overline{J}_\sigma)_{\mu\nu} = \varepsilon_{\sigma\mu\nu}$  are the elements of the algebra of the adjoint representation of  $SU(2)$ . The matrix

$$D(\theta, \phi) = \begin{pmatrix} \cos \theta \cos \phi & -\sin \phi & \sin \theta \cos \phi \\ \cos \theta \sin \phi & \cos \phi & \sin \theta \sin \phi \\ -\sin \theta & 0 & \cos \theta \end{pmatrix} \quad (29)$$

with  $\overline{D}d = 1$ . The nonzero commutation relations of new variables are:

$$[\theta, p_\theta] = [\phi, p_\phi] = 1, \quad [\Phi_i(x), \Pi_j(y)] = A_{ij}(x-y). \quad (30)$$

The operator  $\overline{t}_\alpha$  are the generators of the inverse transformations of  $SU(2)$ , whereas  $t_\alpha$  the generators of direct transformation and

$$t_\alpha = D_{\alpha 2} p_\theta - D_{\alpha 1} \frac{1}{\sin \theta} p_\phi.$$

These two operators are related by equality  $t_\alpha = \overline{D}_{\alpha\beta} t_b$ . the further simplification can be introduced by setting  $N_{ji} = \lambda (\overline{J}_m)_{i3} u(x)$  such that the equality is replaced by

$$\lambda \int dx u^2(x) = 1. \quad (31)$$

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<sup>2</sup>Greek indices take on values 1,2,3 and Latin indices - 1,2.

Besides  $F_{\sigma j} = -\lambda \int dx u(x) \Phi_3(x) = -\lambda h$ . As in the case of  $U(1)$  symmetry we assume the wave functional to be

$$\Psi = (g - \lambda h)^{-1} \tilde{\Psi}.$$

After all this one obtains the following expression for the total Hamiltonian of the model under consideration:

$$H = \int dx \frac{1}{2} \{ \Pi_\alpha(x) \Pi_\alpha(x) + (gu'(x) \delta_{\alpha 3} + \Phi'_\alpha(x)) (gu(x) \delta_{\alpha 3} + \Phi'_\alpha(x)) + U \} \\ \frac{\lambda}{2} (g - \lambda h)^{-1} \{ \bar{t}_j + \bar{T}_j \} (g - \lambda h)^{-1} \{ \bar{t}_j + \bar{T}_j \}. \quad (32)$$

The potential  $U$  is represented as series similar to the case of  $U(1)$  symmetry and we do not give corresponding relation. One can easily verify the correctness of the equality  $T_\alpha = t_\alpha$  and since the commutation relation  $[t_\alpha, \bar{t}_\beta] = 0$  holds the operators  $\bar{t}_j$  can be replaced by c-numbers. The perturbation theory is similar to that of in preceding section. It is evident that the dependence of the energy from the quantum number of isotopic spin is of order  $g^{-2}$  and the energy in this order is proportional to  $j(j+1)$  with integer  $j$ .

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